PERGAMON

International Journal of Solids and Structures 37 (2000) 2035–2058

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Singularity in rotating orthotropic discs and shells

Rajeev Jain[a], K. Ramachandra[a], K.R.Y. Simha[b,*]

[a]*Gas Turbine Research Establishment, Bangalore-560 093, India*
[b]*Department of Mechanical Engineering, Indian Institute of Science, Bangalore-560 012, India*

Received 6 August 1998

Abstract

A unified formulation for studying stresses in rotating polarly orthotropic discs, shallow shells and conical shells is presented. The main focus of this paper is on the examination of singularities when tangential modulus of elasticity (E_θ) is smaller than the radial modulus (E_r). The order of the singularity is extracted by expressing the solutions in terms of modified bessel function with complex argument. The order of the singularity is shown to be ($\sqrt{E_\theta/E_r - 1}$) in all the three cases studied here. There is no singularity present when $E_\theta/E_r \geq 1$. Theoretical results are compared with FEM calculations in all the cases. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Singularity; Orthotropy; Rotating; Plates; Shallow shell; Conical shell

Nomenclature

a	radius of shallow shell
D_θ	$E_\theta h^3/12(1-\nu_{\theta r}\nu_{r\theta})$ flexular rigidity of the plate in tangential direction
D_r	$E_r h^3/12(1-\nu_{\theta r}\nu_{r\theta})$ flexular rigidity of the plate in radial direction
E_θ	Young's modulus of elasticity in tangential direction
E_r	Young's modulus of elasticity in radial direction
h	thickness of the shell
$I_\nu(x)$, $K_\nu(x)$	Modified Bessel's function of νth order and imaginary argument
l	characteristic length $= \sqrt{ah}/\sqrt[4]{12\beta(1-\nu_{\theta r}\nu_{r\theta})}$
$\bar{M}_\theta$	normalised tangential bending moment $= M_\theta/(\rho w^2 r_c^3)$
$\bar{M}_r$	normalised radial bending moment $= M_r/(\rho w^2 r_c^3)$

* Corresponding author. Fax: 00 91 80 3341648/1683.
E-mail address: simha@mecheng.iisc.ernet.in (K.R.Y. Simha).

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PII: S0020-7683(98)00346-1

$\bar{M}_y$	normalised bending moment $= M_y/(\rho w^2 r_c^3)$
M_θ	bending moment per unit length in tangential direction
M_r	bending moment per unit length in radial direction
M_y	bending moment per unit length in conical shell
$\bar{N}_\theta$	normalised tangential membrane force per unit length $= N_\theta/(\rho w^2 r_c^2)$
$\bar{N}_r$	normalised radial membrane force per unit length $= N_r/(\rho w^2 r_c^2)$
$\bar{N}_y$	normalised membrane force per unit length $= N_y/(\rho w^2 r_c^2)$
N_θ	tangential membrane force per unit length
N_r	radial membrane force per unit length in shallow shell
N_y	radial membrane force per unit length in conical-shell
p	load intensity in normal direction $= \rho w^2 h r^2/a^2$
p_r	load intensity in meridional direction $\approx \rho w^2 h r(1 - r^2/2a^2)$
Q_r	shearing force in radial direction
Q_v	vertical shearing force
Q_θ	shearing force in tangential direction
$\bar{r}$	non-dimensional radius $= r/r_c$
w	angular velocity in rad/s
x	nondimensional parameter $= r/l$.

Greek symbols

β	anisotropic parameter $= E_\theta/E_r$
ε_r	radial strain
ε_θ	tangential strain
$\nu_{\theta r}$	Poisson's ratio
$\nu_{r\theta}$	Poisson's ratio
ρ	mass density of material
∇^4	biharmonic operator $= [(\mathrm{d}^2/\mathrm{d}r^2) + (1/r)(\mathrm{d}/\mathrm{d}r)]^2$
Ω	body force potential $= -\int p_r\,\mathrm{d}_r$ for shallow shell, $-\rho w^2 h y^2 \sin^2\alpha/2$ for conical shell

1. Introduction

Modern design of shells for defence and aerospace applications are increasingly based on the use of composite materials. Composite materials are anisotropic. Particularly, orthotropic design of composites is not only practically viable but also analytically more tractable. However, the resulting analysis sometimes leads to singularities in the stress distribution. Singularities arising on account of geometrical and material discontinuities, for instance, sharp corners or a sharp wedge-like inclusion, are well known. Singularities have been extensively studied by researchers in fracture and contact mechanics.

The purpose of this paper is to highlight the existence of singularities in continuous homogeneous rotating orthotropic discs and shells. This feature is not observed in the corresponding isotropic case. Singularities are also not present when the tangential modulus of elasticity in orthotropic discs exceeds the radial modulus. In other words, the anisotropic parameter defined as the ratio of tangential to radial modulus $\beta = E_\theta/E_r$ plays a significant role in this analysis. This aspect was mentioned by Tang (1969) for rotating anisotropic discs. However, the same feature is also common to rotating orthotropic shallow shells and conical shells as described in this paper. As observed by Tang (1969), singularities can be avoided by tailoring the elastic properties radially (Jain et al., 1999), or by varying the thickness of the disc.

Most of the work in centrifugal bodyforce axi-symmetric problems pertain to isotropic materials. Examples of rotating discs and shells of uniform and variable thickness include the work of Flügge (1973) and Timoshenko and Woinowsky-Krieger (1989). The general formulation for an isotropic shell was given by Kraus (1967) for structural applications. Rotating discs and shells constitute a critical segment in the design of gas turbine and compressor discs, nose cones of commercial and military aircraft. Stresses are sensitive to small changes in apical cone angle in rotating conical shells (Meriam, 1943). Extensive work has been done on rotating discs with varying thickness and density. The variable density approach can be used to predict displacements and stresses in rotating shallow shell with variable thickness (Simha et al., 1994). A number of papers on isotropic plates and shells discuss the singularity for concentrated loads (Dundurs and Jahanshahi, 1965; Lukasiewicz, 1967; Sanders, 1970). These investigations differ from the present study on singularities arising from anisotropic material properties.

Güven (1992) investigated the influence of a radial density gradient on elastic-plastic stresses for a linear strain hardening material. Later, Güven (1998) showed that the growth of plastic region decreases significantly upon increasing the values of the thickness parameter. You et al. (1997) determined the stresses and displacement in rotating discs with non-linear strain hardening by assuming a polynomial stress-plastic strain relation using perturbation.

Extensive work has been done on anisotropic rotating disc with varying thickness and density. Murthy and Sherbourne (1970) obtained complete analytical solutions for rotating anisotropic annular discs with variable thickness and a disc mounted on a circular rigid shaft. Later, they (Sherbourne and Murthy, 1974) used dynamic relaxation technique for analysing anisotropic discs with variable profiles. This was followed by a postbuckling analysis of orthotropic plates by Sherbourne and Pandey (1992) giving due recognition for the influence of singularities. The influence of material density on the stresses and displacement of a rotating polar orthotropic circular disc was investigated by Reddy and Srinath (1974) and Chang (1976). Leissa and Vagins (1978) showed how to tailor elastic moduli to eliminate undesirable stress concentrations in annular orthotropic discs. Recently, Galmudi and Dvorkin (1995) highlighted the peculiarities of stress distribution in hollow anisotropic cylinders subjected to external pressure. This work was elaborated further by Horgan and Baxter (1996) for anisotropic rotating disk and spherical shells. Deployable solar panels and antennae for space applications demand light weight construction, and composites may substitute conventional designs in future missions. Regarding ground application, flywheels and large bevel gears represent possible candidates for exploiting composite materials for their construction. The chief attraction for using composites lies in their directional strength and elastic properties that can be tailored by the designer to meet a specific requirement. It is also possible to manufacture composite components with spatially varying properties. Despite all these attractive features, theoretical analyses of rotating discs and shells have been somewhat neglected owing to the widespread use of numerical methods in modern engineering design. Ideally, numerical analysis should supplement and support theoretically derived notions and experimental observation. Singularities demand special theoretical attention before numerical implementation.

Formulation of axi-symmetric problems for orthotropic materials follows the same procedure as for isotropic materials. However, in general, the solution for the governing differential equations becomes much more difficult. Consequently, the analysis is generally confined to shallow shells, conical shells, and, of course, flat plates. Moreover, for shells, the elastic properties and the shell thickness are kept constant. The governing equations are then solved to satisfy the boundary conditions.

Singularities occur in the case of solid discs and shells. This existence of singularities is also demonstrated using a commercial FEM software although the *order* of the singularity is not determined. The singularity order is a very important parameter in engineering design. Theoretical analysis alone can provide the resolution of the order of singularities. Hence, the main focus of this paper is on theoretical analysis of rotating orthotropic discs and shells. With this introduction, Section 2 provides the formulation and result for a rotating orthotropic flat disc. In this section on flat discs, it will be shown that both radial and tangential stresses become infinite at the centre of the disc when $\beta < 1$. Further, the order of the singularity is shown to be $\sqrt{\beta} - 1$. A surprising result for $\beta > 1$ concerns the simultaneous vanishing of radial and tangential stresses at the centre of the disc. For an isotropic disc ($\beta = 1$), it may be recalled that the stresses are finite and equal. Another important issue addressed in this section pertains to the correct choice for the singularity order. It is found that the analysis gives rise to two singularities of which only one is correct. This selection leads to a finite strain energy as will be proved in this section. This preliminary analysis on discs guides further investigation on shells.

The analysis of a rotating shallow shell is presented in Section 3 along with results. As in the case of rotating discs, stresses become singular at the center for $\beta < 1$. Surprisingly, the order of the singularity remains $\sqrt{\beta} - 1$ as in a flat disc. In the limit of an infinite radius of curvature for the shallow shell, results in Section 2 for the flat disc are recovered for stresses.

Finally Section 4 deals with a rotating orthotropic conical shell. In this case also, the results are controlled by β. Thus, in all the three cases, the singularity order is $(\sqrt{\beta} - 1)$.

2. Flat disc formulation

The following analysis is based on anisotropic theory of elasticity with stress-strain relations obeying generalized Hooke's law (Lekhnitskii, 1981). It is assumed that the principal axes of anisotropy coincide with radial and tangential direction of the disc. The formulation used for the flat disc sets the general procedure for shells in the later sections.

The basic differential equation of equilibrium for a rotating disc is

$$\frac{\mathrm{d}(rN_r)}{\mathrm{d}r} - N_\theta + \rho w^2 h r^2 = 0. \tag{1}$$

The strain–displacement relations are

$$\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \end{bmatrix} = \begin{bmatrix} \mathrm{d}u/\mathrm{d}r \\ u/r \end{bmatrix} \tag{2}$$

The strain–compatibility relation is

$$\frac{\mathrm{d}}{\mathrm{d}_r}(r\varepsilon_\theta) = \varepsilon_r \tag{3}$$

The stress–strain relation for orthotropic material (Lekhnitskii, 1981)

$$\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \end{bmatrix} = \frac{1}{h}\begin{bmatrix} 1 & -\nu_{r\theta} \\ -\nu_{\theta r} & 1 \end{bmatrix}\begin{bmatrix} N_r/E_r \\ N_\theta/E_\theta \end{bmatrix} \tag{4}$$

Assuming N_r and N_θ to be given by a function F defined as

$$\begin{bmatrix} N_r \\ N_\theta \end{bmatrix} = \begin{bmatrix} \dfrac{1}{r}\dfrac{\mathrm{d}F}{\mathrm{d}r} + \Omega \\ \dfrac{\mathrm{d}^2F}{\mathrm{d}r^2} + \Omega \end{bmatrix} \tag{5}$$

where $\Omega = -\rho w^2 h r^2/2$ is the body force potential. From eqns (1)–(5) the governing differential equation for F emerges as

$$r\frac{\mathrm{d}^3F}{\mathrm{d}r^3}+\frac{\mathrm{d}^2F}{\mathrm{d}r^2}-\frac{\beta}{r}\frac{\mathrm{d}F}{\mathrm{d}r}=(\beta-1)\Omega+(\nu_{\theta r}-1)r\frac{\mathrm{d}\Omega}{\mathrm{d}r} \tag{6}$$

Differentiating eqn (6) provides a general structure for the governing differential equation as

$$L(F)=\frac{\mathrm{d}^4F}{\mathrm{d}r^4}+\frac{2}{r}\frac{\mathrm{d}^3F}{\mathrm{d}r^3}-\frac{\beta}{r^2}\frac{\mathrm{d}^2F}{\mathrm{d}r^2}+\frac{\beta}{r^3}\frac{\mathrm{d}F}{\mathrm{d}r}=(\beta+\nu_{\theta r}-2)\frac{1}{r}\frac{\mathrm{d}\Omega}{\mathrm{d}r}+(\nu_{\theta r}-1)\frac{\mathrm{d}^2\Omega}{\mathrm{d}r^2} \tag{7}$$

The above form of eqn (7) carries over to shells as described later. Solving eqn (7) will result in the following equation for $\beta\neq 1$

$$F=K_1r^{\sqrt{\beta}+1}+K_2r^{-\sqrt{\beta}+1}+\frac{(3-2\nu_{\theta r}-\beta)}{8(9-\beta)}\rho w^2hr^4+B_1 \tag{8}$$

The membrane stresses obtained from eqn (8) in a rotating disc are

$$\begin{aligned}N_r&=C_1r^{\sqrt{\beta}-1}+C_2r^{-\sqrt{\beta}-1}-\frac{(3+\nu_{\theta r})}{(9-\beta)}\rho w^2hr^2\\ N_\theta&=C_1\sqrt{\beta}r^{\sqrt{\beta}-1}-\sqrt{\beta}C_2r^{-\sqrt{\beta}-1}-\frac{(\beta+3\nu_{\theta r})}{(9-\beta)}\rho w^2hr^2\end{aligned} \tag{9}$$

A non-dimensional form of stresses is obtained upon dividing by $\rho w^2hr_c^2$

$$\begin{aligned}\bar{N}_r&=\bar{C}_1\bar{r}^{\sqrt{\beta}-1}+\bar{C}_2\bar{r}^{-\sqrt{\beta}-1}-\frac{(3+\nu_{\theta r})}{(9-\beta)}\bar{r}^2\\ \bar{N}_\theta&=\bar{C}_1\sqrt{\beta}\bar{r}^{\sqrt{\beta}-1}-\sqrt{\beta}\bar{C}_2\bar{r}^{-}\sqrt{\beta}-1-\frac{(\beta+3\nu_{\theta r})}{(9-\beta)}\bar{r}^2\end{aligned} \tag{10}$$

From the result, it is clear that the order of singularity in the stresses is either $\sqrt{\beta}-1$ or $-(\sqrt{\beta}+1)$. To select the correct answer we examine the strain energy stored in a solid disc of radius r_c. The strain energy is given by

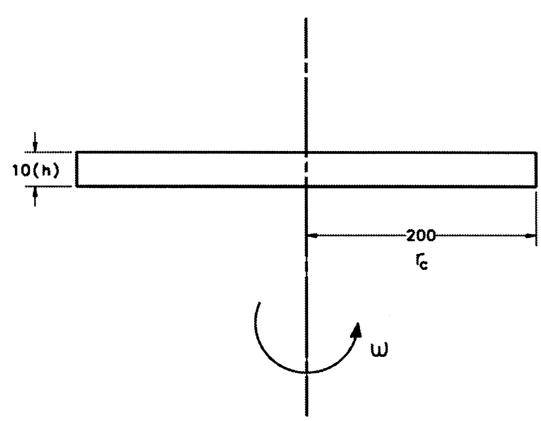


Fig. 1. Rotating anisotropic disc.

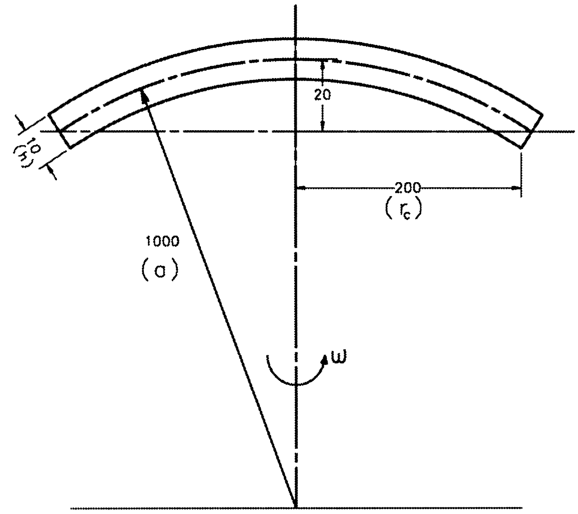


Fig. 2. Rotating shallow shell configuration.

$$U = \frac{1}{h}\int_0^{r_c} \left(N_r^2 + \beta N_\theta^2 - 2N_r N_\theta \nu_{\theta r}\right) 2\pi r \, \mathrm{d}r \tag{11}$$

The singularity of order $-(\sqrt{\beta}+1)$ leads to an infinite amount of strain energy in the disc. This is not permissible. Therefore, the constant C_2 in eqn (9) must be set equal to zero. A similar situation is also encountered in the case of shells. The remaining constant C_1 is determined by applying the boundary condition $N_r = 0$ at $r = r_c$. This leads to $\bar{C}_1 = (3+\nu_{\theta r})/(9-\beta)$.

As a numerical illustration of the above ideas, stresses in a 200 mm radius and 10 mm thick anisotropic disc (Fig. 1) is considered for three values of the anisotropic parameter $\beta = 1/2$, 1 and 2. The anisotropic elastic constants used here are: $E_\theta = 210$ GPa; $\rho = 7800$ kg/m^3; $\nu_{\theta r} = 0.3$. The results are also obtained using a commercial FEM software. Figure 4 shows the variation of $\bar{N}_r$ and $\bar{N}_\theta$ respectively. The formation of the singularity is evident when $\beta = 1/2$. The order of the singularity is $1/\sqrt{2} - 1$. There is no singularity present for $\beta \geqslant 1$. Singularity is observed only when $\beta = 0.5$ at the

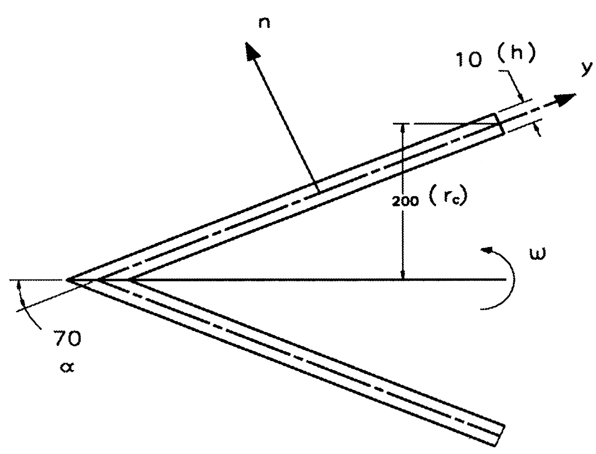


Fig. 3. Rotating conical shell configuration.

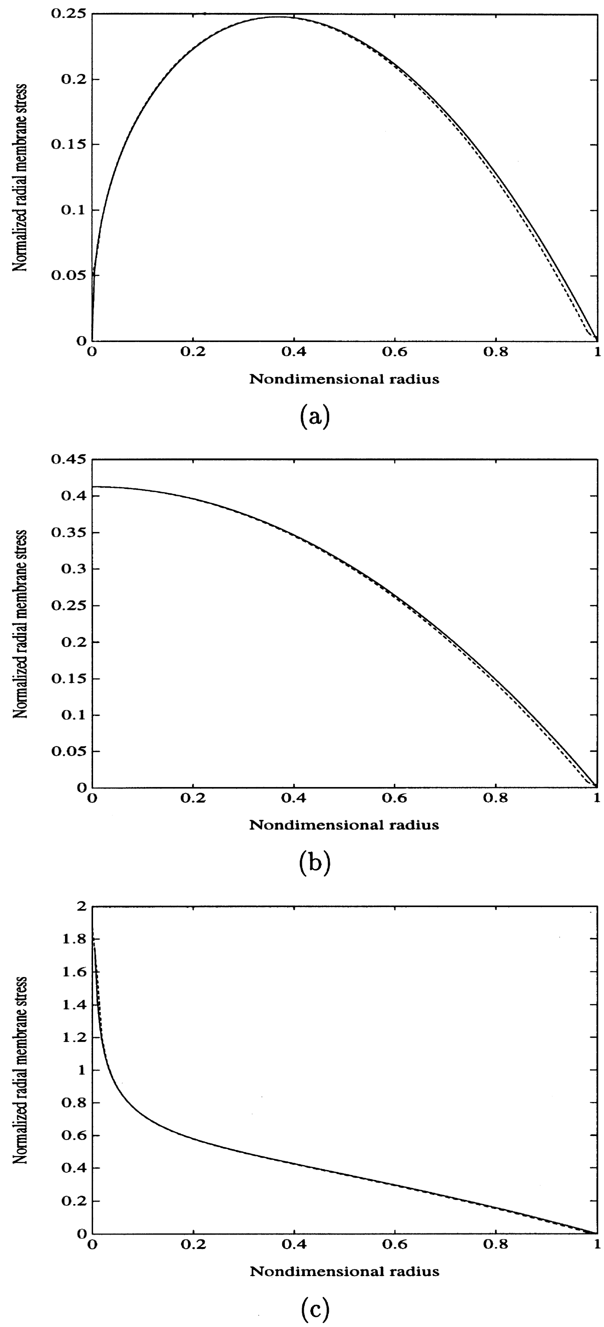


Fig. 4. Comparison of normalised radial stress (solid line) with FEM (dashed line) in rotating flat plate for (a) $\beta = 2$, (b) $\beta = 1$ and (c) $\beta = 0.5$.

centre of the disc. When $\beta = 2$, the centre of the disc is stress free ($\bar{N}_r = \bar{N}_\theta = 0$). The isotropic case ($\beta = 1$) gives equal values for $\bar{N}_r$ and $\bar{N}_\theta$.

3. Shallow shell formulation

We follow the general procedure outlined in Flügge (1973) and Timoshenko and Woinowosky-Krieger (1989).

The differential equations of equilibrium for rotating shallow shell are

$$\frac{\mathrm{d}(rN_r)}{\mathrm{d}r} - N_\theta - \frac{r}{a}Q_r + rp_r = 0 \tag{12}$$

$$\frac{\mathrm{d}(rQ_r)}{\mathrm{d}r} + \frac{r}{a}(N_r + N_\theta) + rp = 0 \tag{13}$$

$$\frac{\mathrm{d}(rM_r)}{\mathrm{d}r} - M_\theta - rQ_r = 0 \tag{14}$$

The strains are

$$\begin{aligned} \varepsilon_r &= \frac{1}{hE_r}(N_r - \nu_{r\theta}N_\theta) = \frac{\mathrm{d}v}{\mathrm{d}r} - \frac{w}{a} \\ \varepsilon_\theta &= \frac{1}{hE_\theta}(N_\theta - \nu_{\theta r}N_r) = \frac{v}{r} - \frac{w}{a} \end{aligned} \tag{15}$$

The bending moments are

$$\begin{aligned} M_r &= -D_r(\chi_r + \nu_{\theta r}\chi_\theta) = -D_r\left(\frac{\mathrm{d}^2w}{\mathrm{d}r^2} + \frac{\nu_{\theta r}}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\right) \\ M_\theta &= -D_\theta(\chi_\theta + \nu_{r\theta}\chi_r) = -D_\theta\left(\frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r} + \nu_{r\theta}\frac{\mathrm{d}^2w}{\mathrm{d}r^2}\right) \end{aligned} \tag{16}$$

Assuming $p_r = -\mathrm{d}\Omega/\mathrm{d}r$, Ω representing a radial body force potential, the force resultants per unit length are

$$\begin{aligned} N_r &= \frac{1}{r}\frac{\mathrm{d}F}{\mathrm{d}r} + \Omega \\ N_\theta &= \frac{\mathrm{d}^2F}{\mathrm{d}r^2} + \Omega \end{aligned} \tag{17}$$

Using eqn (15) the compatibility equation becomes

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\varepsilon_\theta}{\mathrm{d}r}\right) - \frac{1}{r}\frac{\mathrm{d}\varepsilon_r}{\mathrm{d}r} + \frac{1}{a}\nabla^2 w = 0 \tag{18}$$

where ∇^2 is Laplace differential operator

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}.$$

Combining eqns (15) and (17) we arrive at the following fundamental equation for F and w:

$$L(F) + \frac{\beta h E_r}{a}\nabla^2 w = -(1-\nu_{\theta r})\nabla^2\Omega + \frac{(\beta-1)}{r}\frac{\mathrm{d}\Omega}{\mathrm{d}r} \tag{19}$$

The second fundamental relation between F and w is obtained by substituting Q_r from eqn (14) in eqn (13).

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{d}(rM_r)}{\mathrm{d}r} - M_\theta\right] + \frac{r}{a}(N_r + N_\theta) + rp = 0 \tag{20}$$

Using eqns (16) and (17) in combination with eqn (20) gives

$$L(w) - \frac{1}{aD_r}\nabla^2 F = \frac{2\Omega}{aD_r} + \frac{p}{D_r} \tag{21}$$

Equations (19) and (21) can be coupled together by multiplying eqn (19) by $-\lambda$ and adding together yields:

$$L(w-\lambda F) - \frac{\lambda\beta h E_r}{a}\nabla^2\left(w + \frac{a}{\lambda\beta h E_r a D_r}F\right) = \lambda(1-\nu_{\theta r})\nabla^2\Omega + \frac{2\Omega}{aD_r} + \frac{p}{D_r} - \lambda\frac{(\beta-1)}{r}\frac{\mathrm{d}\Omega}{\mathrm{d}r} \tag{22}$$

Stipulating $\lambda = -1/\lambda\beta h E_r D_r$, eqn (22) can be written in terms of $\phi = (w - \lambda F)$ as

$$L(\phi) - \frac{\beta h E_r}{a}\lambda\nabla^2\phi = \lambda(1-\nu_{\theta r})\nabla^2\Omega + \frac{2\Omega}{D_r a} + \frac{p}{D_r} - \lambda\frac{\beta-1}{r}\frac{\mathrm{d}\Omega}{\mathrm{d}r} \tag{23}$$

In terms of $\psi = \mathrm{d}\phi/\mathrm{d}r$ eqn (23) becomes

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\psi}{\mathrm{d}r} - \left(\frac{\beta}{r^2} + \frac{\imath}{l^2}\psi\right) = \lambda\left[(1-\nu_{\theta r})\frac{\mathrm{d}\Omega}{\mathrm{d}r} + (1-\beta)\frac{\Omega}{r}\right] + \frac{2}{r\,\mathrm{d}_r a}\int \Omega r\,\mathrm{d}r + \frac{1}{rD_r}\int pr + \frac{A_2}{r} \tag{24}$$

where $l = \sqrt{ah}/4\sqrt{12\beta(1-\nu_{\theta r}\nu_{r\theta})}$ is a characteristic length.

A homogeneous solution of above equation is (McLachlan, 1955)

$$\psi = A_3 I_\nu\left(i^{1/2}r/l\right) + B_3 K_\nu\left(i^{1/2}r/l\right) \tag{25}$$

where I_ν and K_ν are modified Bessel functions of order ν where $\nu = \sqrt{\beta}$. A_3 and B_3 are complex constants to be determined from the boundary conditions viz $N_r = M_r = 0$ at $r = r_c$. The function I_ν is defined at $r = 0$ whereas K_ν is singular at $r = 0$. Therefore, B_3 and A_2 must be set equal to zero for a shallow shell.

Membrane and bending stresses from eqns (24) and (25) are

$$\begin{aligned} N_r &= -\frac{\beta h E_r l}{a}\left(a_3\frac{bei_\nu x}{x} + b_3\frac{ber_\nu x}{x}\right) - R_1 r^2 - R_2 r^4 - R_3 r^6 - R_4 r^8 + \Omega \\ N_\theta &= -\frac{\beta h E_r l}{a}\left(a_3 bei'_\nu x + b_3 ber'_\nu x\right) - 3R_1 r^2 - 5R_2 r^4 - 7R_3 r^6 - 9R_4 r^8 + \Omega \end{aligned} \tag{26}$$

$$M_r = -D_r\left[\frac{a_3}{l}\left(ber'_\nu x + \nu_{\theta r}\frac{ber_\nu x}{x}\right) - \frac{b_3}{l}\left(bei'_\nu x + \nu_{\theta r}\frac{bei_\nu x}{x}\right) + (3+\nu_{\theta r})Q_1 r^2 + (5+\nu_{\theta r})Q_2 r^4 + (7+\nu_{\theta r})Q_3 r^6 + (9+\nu_{\theta r})Q_4 r^8\right]$$

$$M_\theta = -D_\theta\left[\frac{a_3}{l}\left(\frac{ber_\nu x}{x} + \nu_{\theta r} ber'_\nu x\right) - \frac{b_3}{l}\left(\frac{bei_\nu x}{x} + \nu_{\theta r} bei'_\nu x\right) + (1+3\nu_{r\theta})Q_1 r^2 + (1+5\nu_{r\theta})Q_2 r^4 + (1+7\nu_{r\theta})Q_3 r^6 + (1+9\nu_{r\theta})Q_4 r^8\right] \tag{27}$$

where $x = r/l$ is dimensionless parameter, constants Q_i and R_i are

$$Q_1 = 0$$

$$R_1 = \frac{3-\beta-2\nu_{\theta r}}{(9-\beta)}\Omega$$

$$Q_2 = \frac{2(3+\nu_{\theta r})}{(9-\beta)(25-\beta)}\frac{\Omega}{aD_r}$$

$$R_2 = \frac{5-\beta-4\nu_{\theta r}}{(25-\beta)}\frac{\Omega}{4a^2}$$

$$Q_3 = -\frac{5+\beta+6\nu_{\theta r}}{(25-\beta)(49-\beta)}\frac{\Omega}{6D_r a^3}$$

$$R_3 = -\frac{(7-\beta-6\nu_{\theta r})}{24(49-\beta)}\frac{\Omega}{a^4} + \frac{\beta h E_r Q_2}{a(49-\beta)}$$

$$Q_4 = \frac{1}{(81-\beta)}\left(\frac{\Omega}{96 D_r a^5} - \frac{R_3}{aD_r}\right)$$

$$R_4 = -1(1-\beta)\frac{\Omega}{64a^6} - (1-\nu_{\theta r})\frac{\Omega}{8a^6} + \frac{\beta h E_r}{a}Q_3$$

4. Conical shell formulation

We follow the general procedure outlined in Flügge (1973) and Timoshenko and Woinowosky-Krieger (1989).

The differential equations of equilibrium for conical shell are

$$\frac{d(yN_y)}{dy} - N_\theta + \rho w^2 h y^2 \sin^2\alpha = 0 \tag{28}$$

$$\frac{\mathrm{d}(yQ_y)}{\mathrm{d}y} + N_\theta \cot\alpha - \rho w^2 h y^2 \sin\alpha\cos\alpha = 0 \tag{29}$$

$$\frac{d(yM_y)}{\mathrm{d}y} - M_\theta - yQ_y = 0 \tag{30}$$

The strains are

$$\varepsilon_y = \frac{1}{hE_y}\left(N_y - \nu_{y\theta}N_\theta\right) = \frac{\mathrm{d}v}{\mathrm{d}l}$$

$$\varepsilon_\theta = \frac{1}{hE_\theta}\left(N_\theta - \nu_{\theta y}N_\theta\right) = \frac{v}{y} - \frac{w\cot\alpha}{a}. \tag{31}$$

The bending moments are

$$M_y = -D_y\left(\chi_y + \nu\chi_\theta\right) = -D_y\left(\frac{\mathrm{d}^2w}{\mathrm{d}y^2} + \frac{\nu_{\theta y}}{y}\frac{\mathrm{d}w}{\mathrm{d}y}\right)$$

$$M_\theta = -D_\theta\left(\chi_\theta + \nu\chi_y\right) = -D_\theta\left(\frac{1}{y}\frac{\mathrm{d}w}{\mathrm{d}y} + \nu_{y\theta}\frac{\mathrm{d}^2w}{\mathrm{d}y^2}\right). \tag{32}$$

The force resultants are

$$N_y = \frac{1}{y}\frac{\mathrm{d}F}{\mathrm{d}y} + \Omega$$

$$N_\theta = \frac{\mathrm{d}^2F}{\mathrm{d}y^2} + \Omega \tag{33}$$

where $\Omega = -\rho w^2 h y^2 \sin^2\alpha/2$ represents body force potential.

Using eqn (31) the compatibility equation becomes

$$y\frac{\mathrm{d}^2\varepsilon_\theta}{\mathrm{d}y^2} + 2\frac{\mathrm{d}\varepsilon_\theta}{\mathrm{d}y} - \frac{\mathrm{d}\varepsilon_y}{\mathrm{d}y} + \frac{\mathrm{d}^2w}{\mathrm{d}y^2}\cot\alpha = 0 \tag{34}$$

Combining eqns (31) and (33) in conjunction with eqn (34) we obtain the first fundamental equation for F and w:

$$\frac{\mathrm{d}^4F}{\mathrm{d}y^4} + \frac{2}{y}\frac{\mathrm{d}^3F}{\mathrm{d}y^3} - \frac{\beta}{y^2}\frac{\mathrm{d}^2F}{\mathrm{d}y^2} + \frac{\beta}{y^3}\frac{\mathrm{d}F}{\mathrm{d}y} + \cot\alpha hE_\theta\frac{1}{y}\frac{\mathrm{d}^2w}{\mathrm{d}y^2} = \left(\nu_{\theta y} - 1\right)\frac{\mathrm{d}^2\Omega}{\mathrm{d}y^2} + \left(\beta + \nu_{\theta y} - 2\right)\frac{\mathrm{d}\Omega}{\mathrm{d}y} \tag{35}$$

The second fundamental relation between F and w is obtained by substituting yQ_y from eqn (30) in eqn (29).

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{\mathrm{d}(yM_y)}{\mathrm{d}y} - M_\theta\right] + N_\theta\cot\alpha - \rho w^2 h y^2 \sin\alpha\cos\alpha = 0 \tag{36}$$

Using eqns (32) and (33) in combination with eqn (36) gives

$$\frac{d^4 w}{dy^4} + \frac{2}{y}\frac{d^3 w}{dy^3} - \frac{\beta}{y^2}\frac{d^2 w}{dy^2} + \frac{\beta}{y^3}\frac{dw}{dy} - \frac{\cot\alpha}{yD_y}\frac{d^2 F}{dy^2} = \frac{\cot\alpha}{yD_y}\Omega - \frac{\rho w^2 hy \sin\alpha\cos\alpha}{D_y} \quad (37)$$

Multiplying eqn (35) by $-\lambda$ and adding eqns (35) and (37) yields:

$$L(w - \lambda F) - \frac{\lambda h E_\theta \cot\alpha}{y}\left(\frac{d^2 w}{dy^2} + \frac{1}{hE_{\theta y} D_y \lambda}\frac{d^2 F}{dy^2}\right) = \left(2\nu_{\theta y} + \beta - 3\right)\rho w^2 h \sin^2\alpha\lambda - \frac{3\rho w^2 h \sin\alpha\cos\alpha}{2D_y} y \quad (38)$$

where L was defined earlier in eqn (7)

$$L = \frac{d^4}{dy^4}(\cdots) + \frac{2}{y}\frac{d^3}{dy^3}(\cdots) - \frac{\beta}{y^2}\frac{d^2}{dy^2}(\cdots) + \frac{\beta}{y^3}\frac{d}{dy}(\cdots)$$

Stipulating $-\lambda = 1/\lambda h E_\theta D_y$ and defining $\phi = w - \lambda F$ then eqn (38) becomes

$$L(\phi) - \frac{l\mu_c}{y}\frac{d^2\phi}{dy^2} = \left(2\nu_{\theta y} + \beta - 3\right)\rho w^2 h \sin^2\alpha\lambda - \rho w^2 h \sin\alpha\cos\alpha\frac{3y}{2D_y} \quad (39)$$

where $\mu_c = \sqrt{12(1 - \nu_{\theta y}\nu_{y\theta})\beta}\cot\alpha/h$.

In terms of $\psi = d\phi/dy$, eqn (39) can be simplified to

$$\frac{d^2\psi}{dy^2} + \frac{1}{y}\frac{d\psi}{dy} - \left(\frac{\beta}{y^2} + \frac{i\mu_c}{y}\right)\psi = \left(2\nu_{\theta y} + \beta - 3\right)\rho w^2 h \sin^2\alpha\frac{y}{2}\lambda - \rho w^2 h \sin\alpha\cos\alpha\frac{y^2}{2D_y} + \frac{A_2}{y} \quad (40)$$

Finally, substituting $\eta = 2\sqrt{\mu_c y}$ gives

$$\frac{d^2\psi}{d\eta^2} + \frac{1}{\eta}\frac{d\psi}{d\eta} - \left(\frac{4\beta}{\eta^2} + i\right)\psi = f(\eta) \quad (41)$$

where

$$f(\eta) = \left(2\nu_{\theta y} + \beta - 3\right)\rho w^2 h \sin^2\alpha\frac{\eta^4}{32\mu_c^3}\lambda - \rho w^2 h \sin\alpha\cos\alpha\frac{\eta^6}{128 D_y \mu_c^4} + \frac{A_2}{\eta^2}$$

Following, Hildebrand (1962), the complete solution of eqn (41) is

$$\psi = I_\nu\left[\int^\eta \xi f(\xi) K_\nu(\xi)\, d\xi\right] + K_\nu\left[\int^\eta \xi f(\xi) I_\nu(\xi)\, d\xi\right] + A_{c3} I_\nu(\eta) + B_{c3} K_\nu(\eta) \quad (42)$$

I_ν and K_ν are modified Bessel functions of order ν where $\nu = \sqrt{4\beta}$. A_{c3} and B_{c3} are complex constants to be determined from the boundary condition. The function I_ν is defined at $\eta = 0$, whereas $K_\nu(\eta)$ is singular at $\eta = 0$. Therefore, B_{c3} and A_2 must be set equal to zero for a conical shell.

The membrane and bending stresses are

$$N_y = -\tan\alpha\left(4\mu_c^2 D_y\right)\left(a_3\frac{ber_\nu \eta}{\eta^2} + b_3\frac{bei_\nu \eta}{\eta^2}\right) + \mathscr{S}\left(\psi_p\right) + \Omega$$

$$N_\theta = -2\mu_c\tan\alpha\left(\mu_c D_y\right)\left(a_3\frac{ber'_\nu \eta}{\eta} + b_3\frac{bei'_\nu \eta}{\eta}\right) + \mathscr{S}\left(\frac{d\psi_p}{dy}\right) + \Omega \quad (43)$$

$$M_y = -2\mu_c D_y\left[a_3\left(\frac{ber'_v\eta}{\eta} + 2v_{\theta y}\frac{ber_v\eta}{\eta^2}\right) - b_3\left(\frac{bei'_v\eta}{\eta} + 2v_{\theta y}\frac{bei_v\eta}{\eta^2}\right) + \mathscr{R}\left(\psi_p + \frac{d\psi_p}{dy}\right)\right]$$

$$M_\theta = -D_\theta(2\mu_c)\left[a_3\left(\frac{2ber_v\eta}{\eta^2} + v_{\theta y}\frac{ber'_v\eta}{\eta}\right) - b_3\left(\frac{bei_v\eta}{\eta^2} + v_{y\theta}\frac{bei'_v\eta}{\eta}\right) + \mathscr{R}\left(\psi_p + \frac{d\psi_p}{dy}\right)\right] \quad (44)$$

where $\mathscr{R}$ and $\mathscr{S}$ represent the real and imaginary part of

$$\psi_p = I_v\left(\int^\eta \xi f(\xi)K_v(\xi)\, d\xi\right) + K_v\left(\int^\eta \xi f(\xi)I_v(\xi)\, d\xi\right).$$

The order of the singularity is controlled by $I_v(\eta)$ for $\beta < 1$. The function $I_v(\eta)$ can be expanded as

$$I_v(\eta) = \sum_{r=0}^{r=\infty}\frac{\left(\frac{\eta}{2}\right)^{v+2r}}{\Gamma_{v+r+1}r!} = \frac{\eta^v}{2^v\Gamma_{v+1}} + \frac{\eta^{v+2}}{2^{v+2}\Gamma_{v+2}} + \cdots \quad (45)$$

The singularity is now controlled by terms in eqn (43) for the expansions of $ber_v\eta$ and $bei_\beta\eta$. In order to bring out the singularity order, N_y is examined as follows

$$N_y \sim \frac{ber_v\eta}{\eta^2} \sim \frac{\eta^v}{\eta^2} \sim \eta^2\sqrt{\beta} - 2 \sim y\sqrt{\beta} - 1 \quad (46)$$

Once again when, $\beta < 1$, the stresses become singular as in the previous case. It is interesting to note that the order of the singularity is again $\sqrt{\beta} - 1$ as in the previous case. An unexpected result in the case of conical shells is the development of a state of compression near the apex for cone angles below 85°. However, it should be recalled the shell theories do not give correct results near the apex. Therefore, the singularity analysis presented here gives only qualitative representation. For larger cone angles approaching 90° the stresses predicted by the conical shell analysis become tensile, and finally merge with flat plat results for $\alpha = 90°$. The outer apex of the conical shell is unlikely to be a region of stress singularity, but the inner apex of the conical shell could lead to singularities. This issue lies outside the scope of this investigation.

A closed form solution is possible when $\beta = 1$ or 4. For these special cases the stresses and moments are

$$N_y = -\tan\alpha\left(4\mu_c^2 D_y\right)\left(a_3\frac{ber_v\eta}{\eta^2} + b_3\frac{bei_v\eta}{\eta^2}\right) - 4\mu_c^2\tan\alpha D_y\left(S_2 + S_4\eta^4\right) + \Omega$$

$$N_\theta = -2\mu_c\tan\alpha\left(\mu_c D_y\right)\left(a_3\frac{ber'_v\eta}{\eta} + b_3\frac{bei'_v\eta}{\eta}\right) - 2\mu_c^2\tan\alpha D_y\left(2S_2 + 6S_4\eta^4\right) + \Omega \quad (47)$$

$$M_y = -2\mu_c D_y\left[a_3\left(\frac{ber'_v\eta}{\eta} + 2v_{\theta y}\frac{ber_v\eta}{\eta^2}\right) - b_3\left(\frac{bei'_v\eta}{\eta} + 2v_{\theta y}\frac{bei_v\eta}{\eta^2}\right) + 2\left(2 + v_{\theta y}\right)T_3\eta^3\right]$$

$$M_\theta = -D_\theta(2\mu_c)\left[a_3\left(\frac{2ber_v\eta}{\eta^2} + v_{\theta y}\frac{ber'_v\eta}{\eta}\right) - b_3\left(\frac{bei_v\eta}{\eta^2} + v_{y\theta}\frac{bei'_v\eta}{\eta}\right) + 2\left(1 + 2v_{y\theta}\right)T_3\eta^2\right] \quad (48)$$

where

$$S_2 = \frac{(4-\beta)k \tan \alpha}{4\mu_c^4 D_y}$$

$$S_4 = -\frac{k \tan \alpha}{128(3+\nu_{\theta y})\mu_c^4 D_y}$$

$$T_3 = -\frac{k \tan \alpha}{16\mu_c^4 D_y}$$

and $k = (3+\nu_{\theta y})\rho w^2 h \cos^2 \alpha$.

5. Results and discussions

The main results from this investigation are displayed in Figs 4 and 5, 6–9 and 10–13 for rotating anisotropic disc, shallow shell and conical shell, respectively. The formation of singularity is clearly evident in all the cases when $\beta = 1/2$. The order of this singularity is $(1/\sqrt{2} - 1)$. No singularity is present for $\beta \geqslant 1$. For the case of a conical shell of semiapical angle 70°, the membrane stresses are compressive in the apex region as shown in Figs 10–13. This feature of compressive membrane stresses persists for a large range of cone angles. Calculations not included here show that tensile membrane stresses begin to appear for cone angles larger than 85° depending upon the shell thickness. Singularities are also detected by FEM although establishing the singularity order is difficult. In general, FEM results compare well with theoretical results except near the singularity. Establishing the correct order of the singularity is important in performing strain energy calculations.

Rotating discs and shells present many interesting features when the material is orthotropic. Results for isotropic plates and shells are widely known and have been discussed extensively in the literature. Anisotropic discs have also been studied, but a detailed discussion on singularities has not been earlier presented. This paper initiates such a discussion for rotating orthotropic discs and shells: A key finding in this work relates to the order of the singularity which remains the same in all the cases. Further, the singularity order depends only on the anisotropic parameter $\beta = E_\theta/E_r$. For $\beta \geqslant 1$, there is no singularity. For $\beta < 1$, a singularity of order $(\sqrt{\beta} - 1)$ is created at the center of rotating discs and shells. This singularity could lead to dangerous conditions at the center by promoting growth of defects in the form of cavities and cracks.

Holes can grow in rotating anisotropic plates and shells under special conditions. For hole growth to occur, sufficient energy must be released from the original configuration of a solid disc. This appears to be possible when $\beta < 1$ because of singular stresses getting relieved at the center of the disc. For $\beta > 1$, the center of the disc is a stress free region. In the case of conical shell the outer apex region could be in a state of compression though the inner apex region is under tension. This issues lies outside the scope of this paper which is based on a conventional shell theory.

6. Conclusions

Analysis of singularities in rotating discs and shells was described in this paper. A unified mathematical formulation is possible for rotating orthotropic discs, shallow shells and conical shells. This formulation is valid within the approximation of thin plate and shell theories developed for isotropic materials. Further, this formulation can be extended to non-axisymmetric problems. It is also

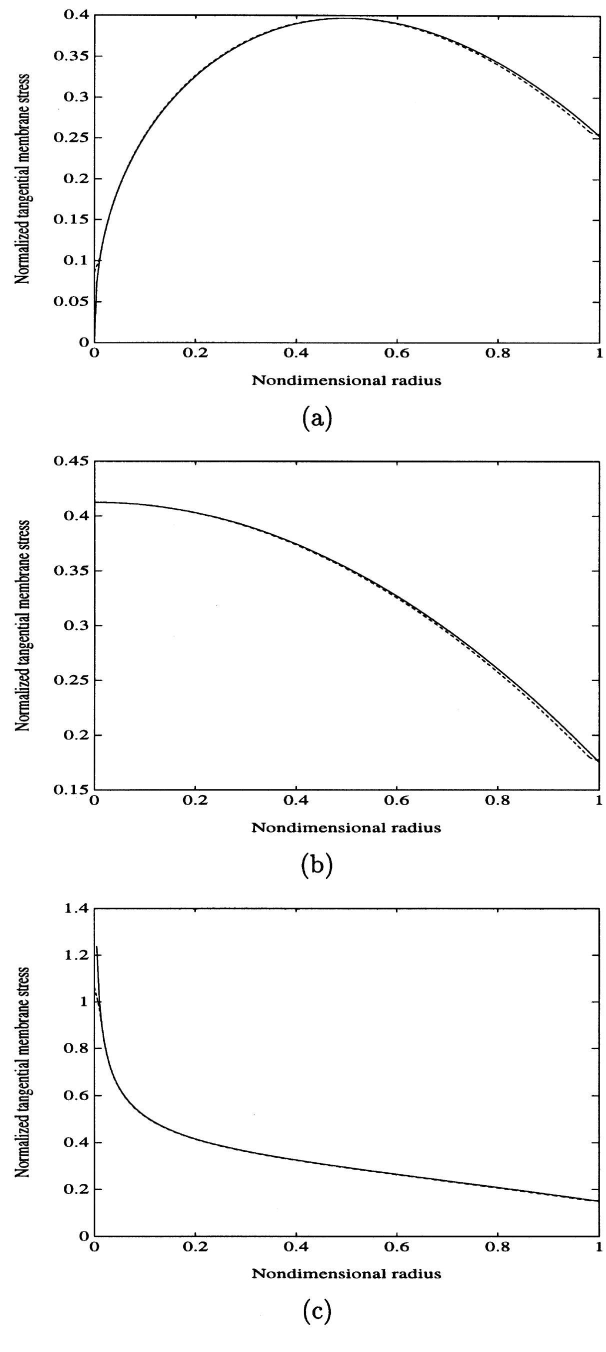


Fig. 5. Comparison of normalised tangential stress (solid line) with FEM (dashed line) in rotating flat plate for (a) $\beta = 2$, (b) $\beta = 1$ and (c) $\beta = 0.5$.

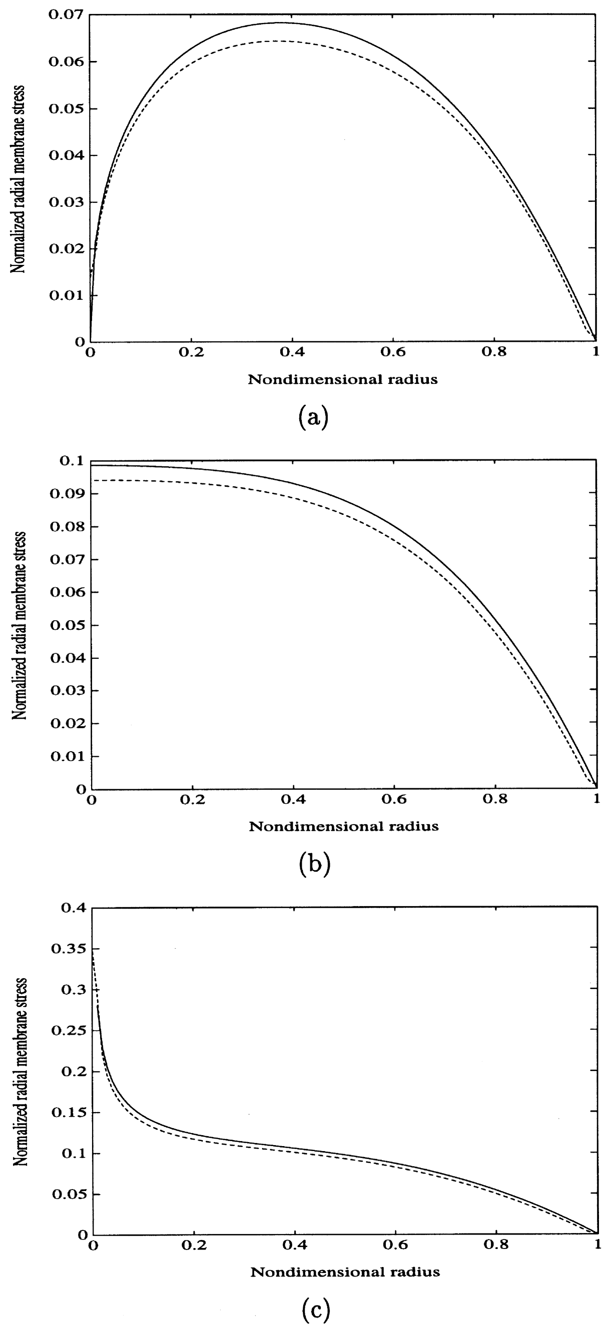
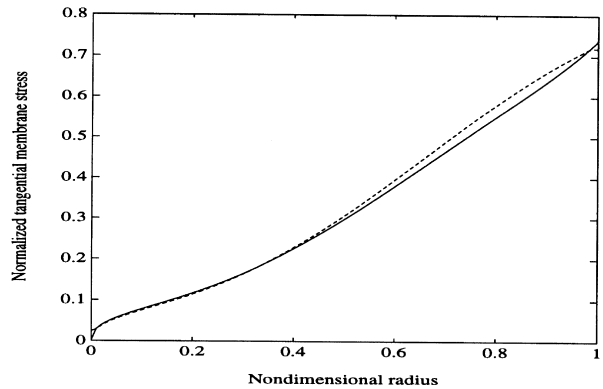
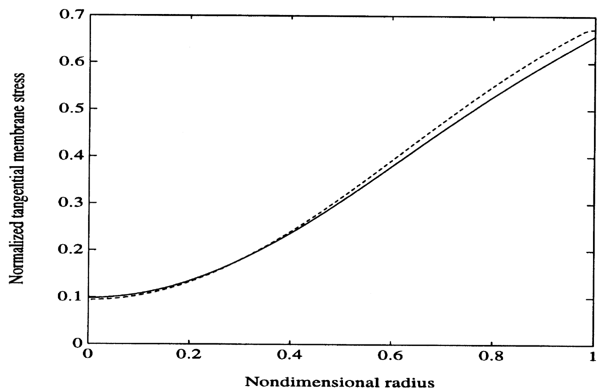


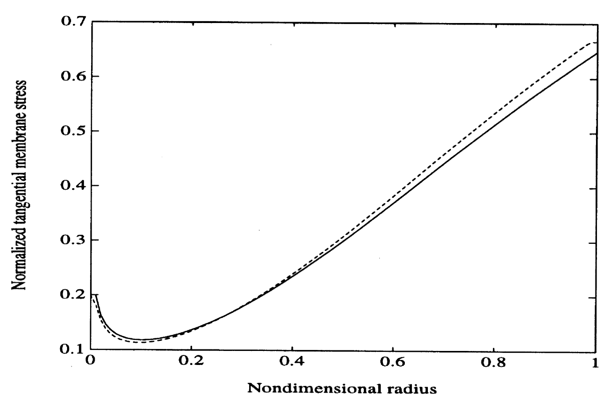
Fig. 6. Comparison of normalised membrane radial stress distribution between analytical (solid line) and FEM (dashed line) results in rotating spherical shell for (a) $\beta = 2$, (b) $\beta = 1$ and (c) $\beta = 0.5$.



(a)



(b)



(c)

Fig. 7. Comparison of normalised membrane tangential stress distribution between analytical (solid line) and FEM (dashed line) results in rotating spherical shell for (a) $\beta = 2$, (b) $\beta = 1$ and (c) $\beta = 0.5$.

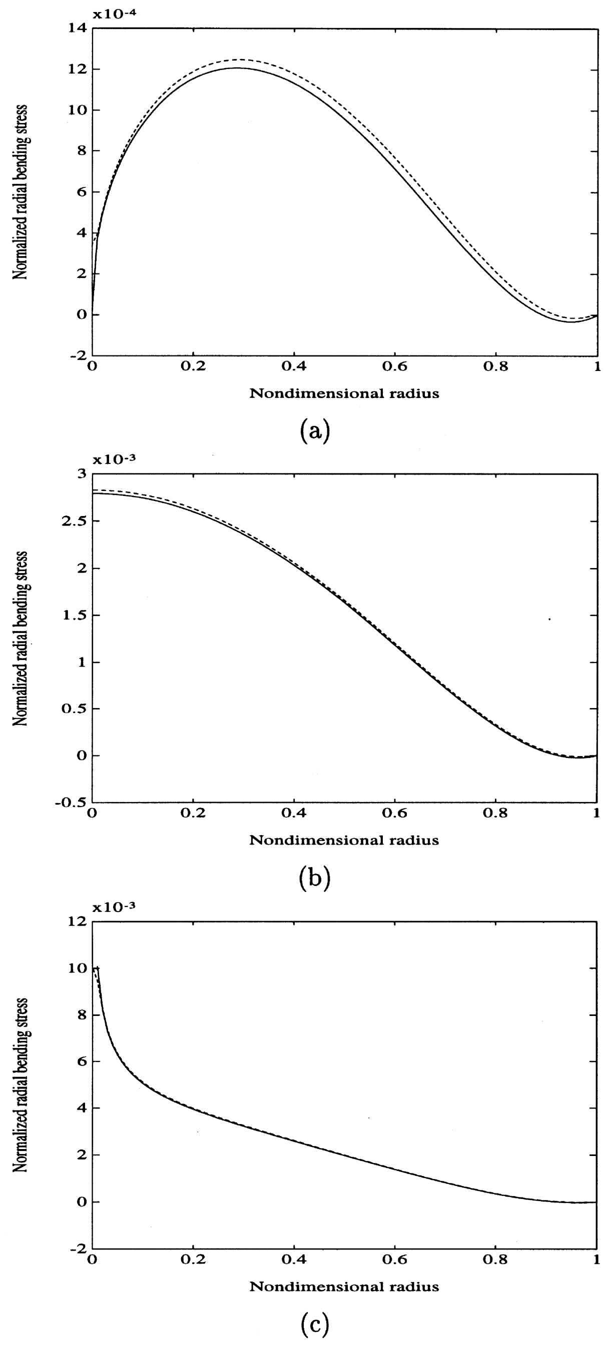


Fig. 8. Comparison of normalised bending radial stress distribution between analytical (solid line) and FEM (dashed line) results in rotating spherical shell for (a) $\beta = 2$, (b) $\beta = 1$ and (c) $\beta = 0.5$.

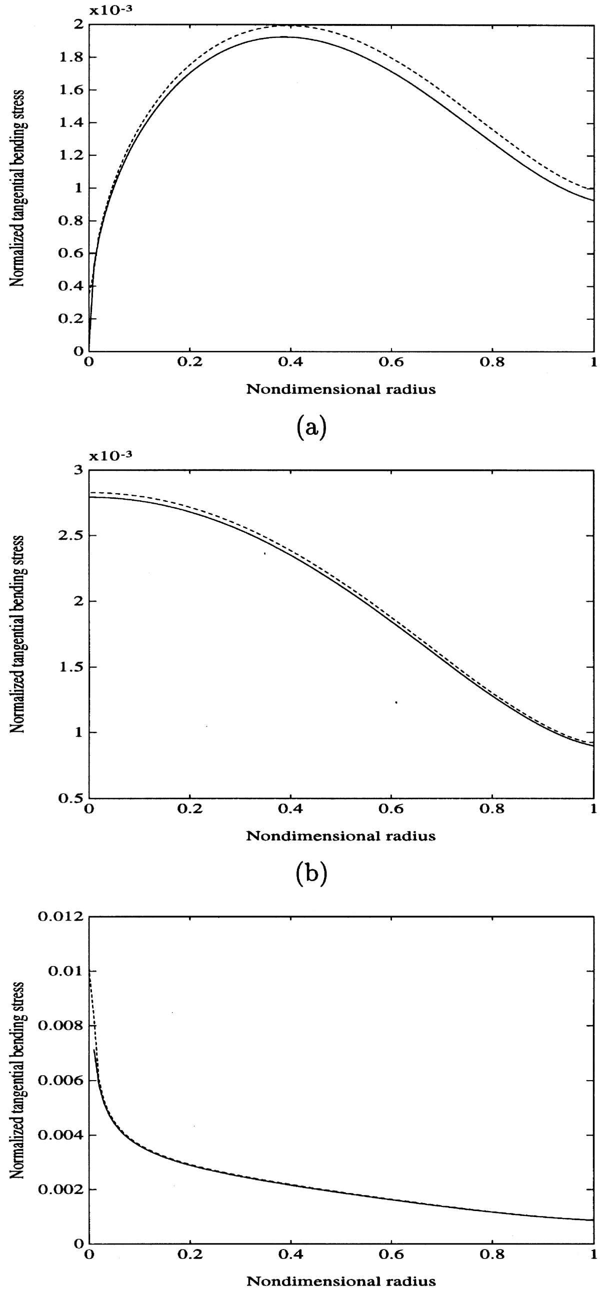


Fig. 9. Comparison of normalised bending tangential stress distribution between analytical (solid line) and FEM (dashed line) results in rotating spherical shell for (a) $\beta = 2$, (b) $\beta = 1$ and (c) $\beta = 0.5$.

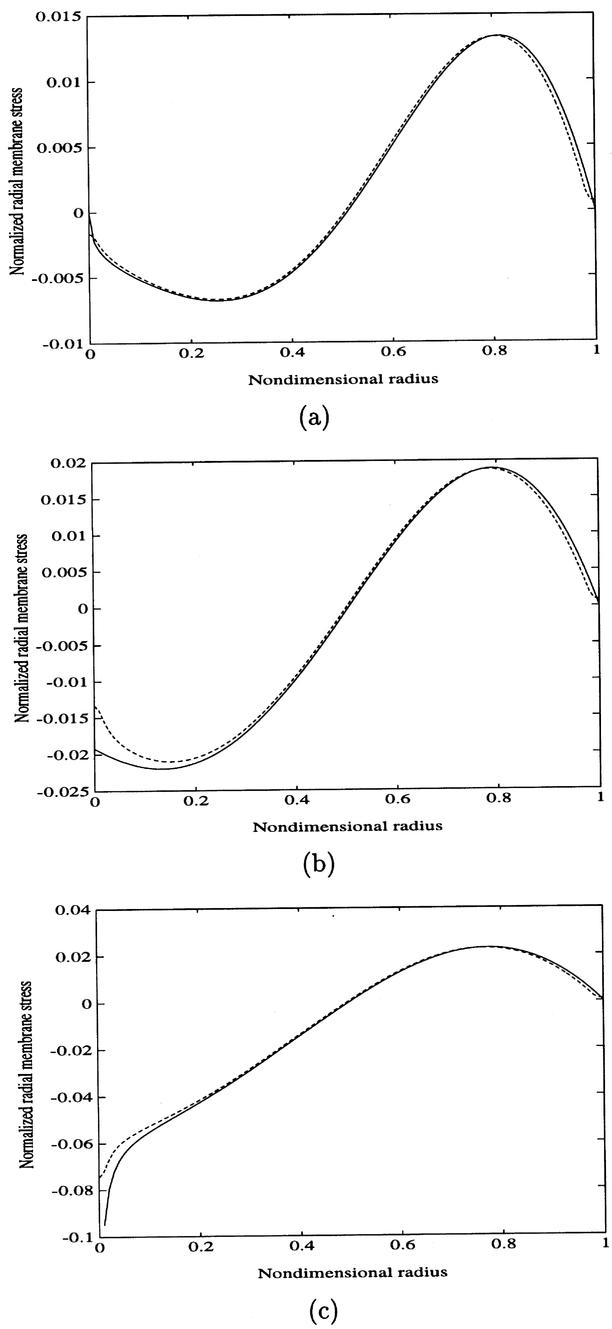


Fig. 10. Comparison of normalised membrane radial stress distribution between analytical (solid line) and FEM (dashed line) results in rotating conical shell of semiapical angle $\alpha = 70°$: (a) $\beta = 2.0$, (b) $\beta = 1$ and (c) $\beta = 0.5$.

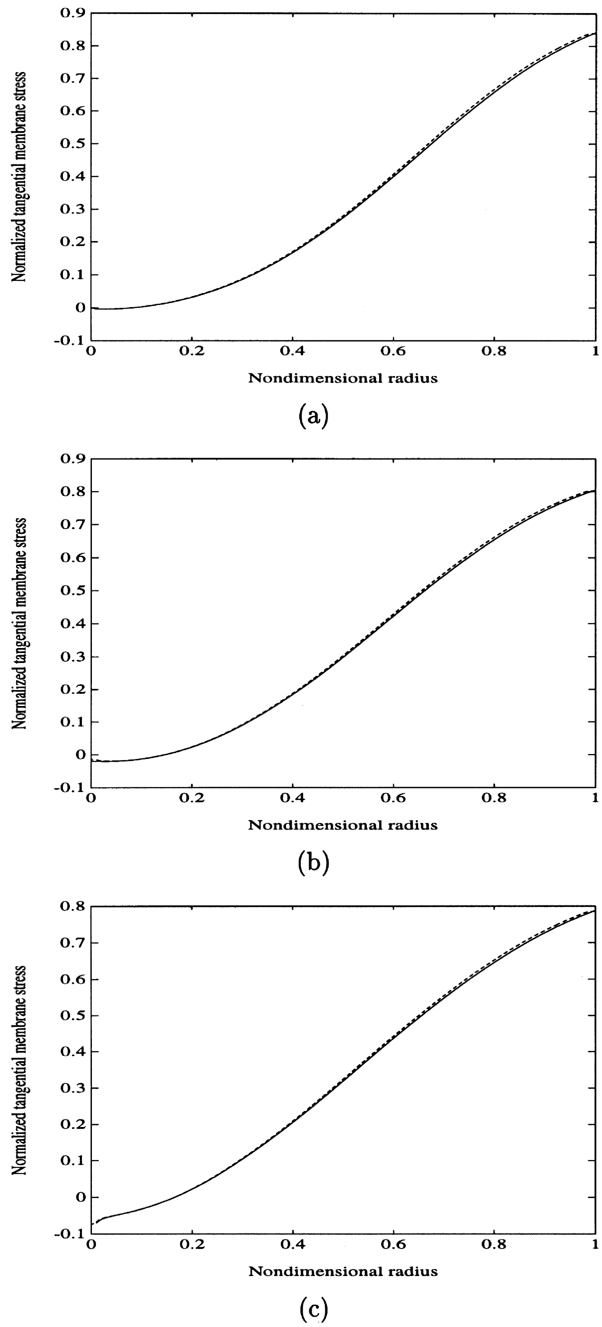


Fig. 11. Comparison of normalised membrane tangential stress distribution between analytical (solid line) and FEM (dashed line) results in rotating conical shell of semiapical angle $\alpha = 70^{\circ}$: (a) $\beta = 2.0$, (b) $\beta = 1$ and (c) $\beta = 0.5$.

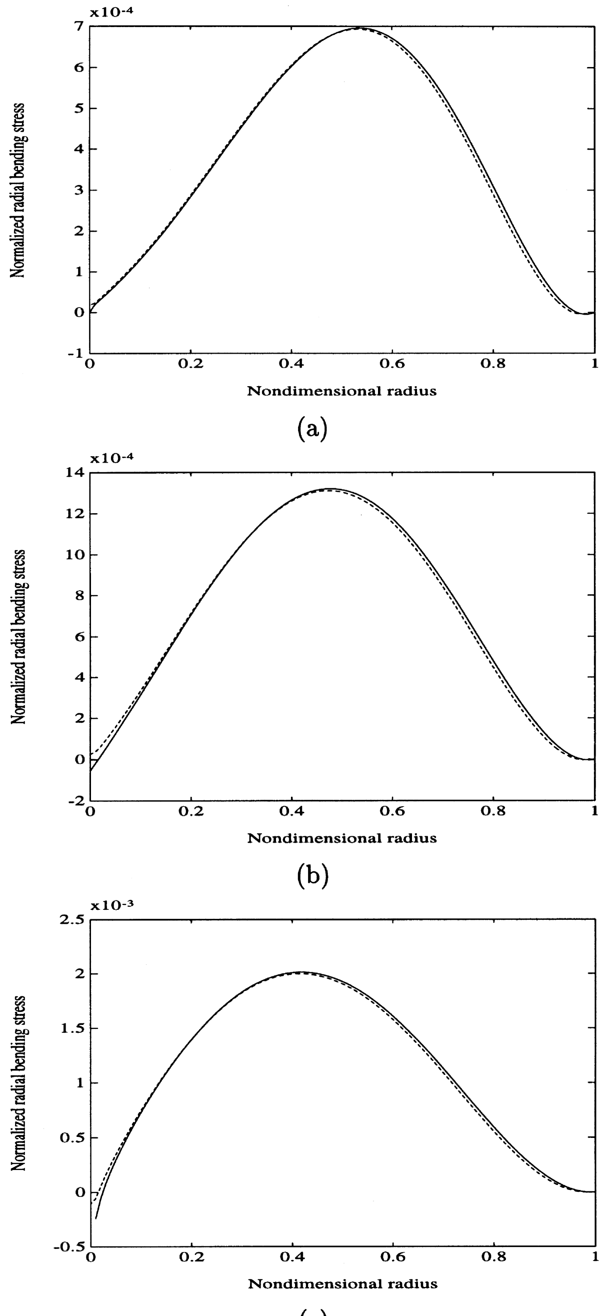


Fig. 12. Comparison of normalised bending radial stress distribution between analytical (solid line) and FEM (dashed line) results in rotating conical shell of semiapical angle $\alpha = 70^{\circ}$: (a) $\beta = 2.0$, (b) $\beta = 1$ and (c) $\beta = 0.5$.

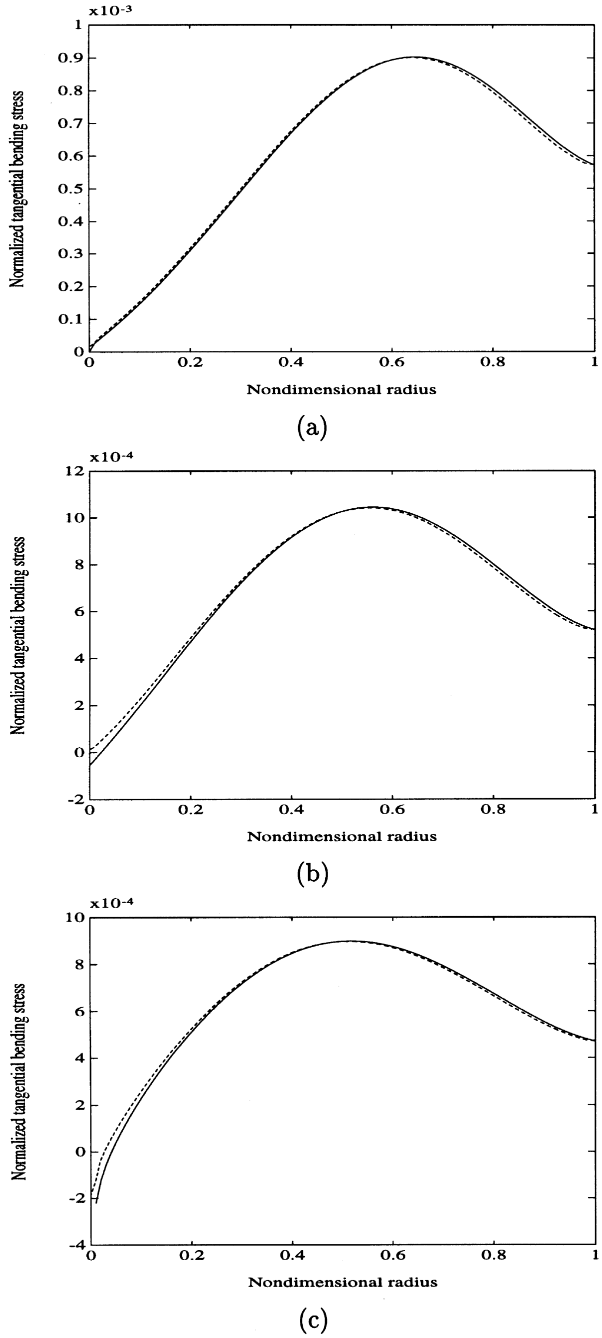


Fig. 13. Comparison of normalised bending tangential stress distribution between analytical (solid line) and FEM (dashed line) results in rotating conical shell of semiapical angle $\alpha = 70°$: (a) $\beta = 2.0$, (b) $\beta = 1$ and (c) $\beta = 0.5$.

possible to obtain results for discs and shells with a central opening. In the case of a central opening, however, both terms in the homogeneous solution should be retained to satisfy the boundary conditions. As a consequence, singularities are not present in this case. Singularities are generated only on account of anisotropy. The existence of such singularities was also illustrated using FEM. Results and discussion presented in this paper highlight design problems that may be encountered in the use of composite materials for rotating components. Singularity analysis needs further attention with regard to both physics and mechanics.

Acknowledgements

The authors are thankful to Mr V Sundararajan, Director, Gas Turbine Research Establishment for providing permission to publish this paper and Mrs N. Leela, Section Head SWE for her support and encouragement. One of the authors (R.J.) is thankful to Mrs Anjana Jain for helping him in preparing this manuscript.

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